Directed graphs

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http://www.4shared.com/file/43395119/3907952d/lect08.html

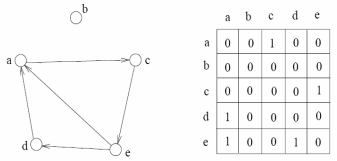
Terminology

- Connected graph
  - A graph is connected if and only if there exists a path between every pair of distinct vertices
- Sub-graph
  - A graph with the vertex and edge set being subsets of the original graph
- Connected Components
  - A connected component of a graph is a maximally connected subgraph of a graph
- Cycle
  - A path in a graph that starts and ends at the same vertex
- Tree
  - A graph G is a tree if and only if it is connected and acyclic
- Directed Graph
  - A graph whose the edges (arcs) are directional
- Directed Acyclic Graph
  - A directed graph with no directed cycles

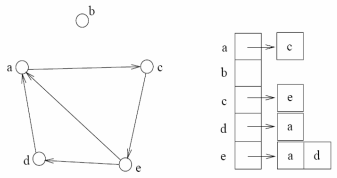
Directed Graphs

- A directed graph can be represented by an adjacency matrix/list the same way as in undirected graph, except:
  - An arc (u, v) only contributes to 1 entry in the adj. matrix or 1 node in the adj. list

1. *Adjacency Matrix*

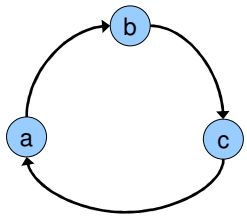


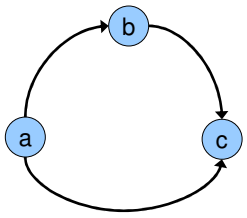
2. *Adjacency List*



Paths/Cycles

- A directed graph can also contain paths and cycles ("directed paths" and "directed cycles")
  - Graph on top has directed paths and directed cycle
  - Graph on bottom has directed paths but NO directed cycle (acyclic)





Graph traversal

- BFS and DFS can be used to traverse a directed graph, the same way as in undirected graph
- To check for connectivity of a graph
  - run BFS or DFS using an arbitrary vertex as the source. If all vertices have been visited, then the graph is connected; otherwise, the graph is disconnected

Finding Connected Components

- Run DFS or BFS from a vertex
  - the set of visited vertices form a connected component
- Find another vertex i which has not been visited before, run DFS or BFS from it
  - we have another connected component
- Repeat the steps until all vertices are visited
- Running time is

$$\sum_i O(n_i + m_i) = O(\sum_i n_i + \sum_i m_i) = O(n + m)$$

A complete graph API

- In the current graph API, only the edges are managed. Therefore we can not know how many vertices there are in the graph. Each vertex need also a name for identification.
- Redefine the graph structure in order the vertices data are stored in a tree as the following

```
typedef struct {
   JRB edges;
   JRB vertices;
} Graph;
```

Quiz 1

- Rewrite the (directed) graph API based the new data structure with the functions below

```
Graph createGraph();
void addVertex(Graph graph, int id, char* name);
char *getVertex(Graph graph, int id);
void addEdge(Graph graph, int v1, int v2);
void hasEdge(Graph graph, int v1, int v2);
int indegree(Graph graph, int v, int* output);
int outdegree(Graph graph, int v, int* output);
int getComponents(Graph graph);
void dropGraph(Graph graph);
```

Topological Sort

- A topological sort is an ordering of vertices in a DAG such that if there is a path from w_i to w_j, then w_j appears after w_i in the ordering.
  - Note, this does **not** mean that if a vertex appears after another in the ordering there is a path between those vertices.
  - Note that a topological sort is not possible if there are cycles.
  - Note also that the ordering is not necessarily unique – any legal ordering will do.

Topological Sort

- One can make use of the direction in the directed graph to represent a dependent relationship
  - COMP104 is a pre-requisite of COMP171
  - Breakfast has to be taken before lunch
- A typical application is to schedule an order preserving the order-of-completion constraints following a topological sort algorithm
  - We let each vertex represents a task to be executed. Tasks are inter-dependent that some tasks cannot start before another task finishes
  - Given a directed acyclic graph, our goal is to output a linear order of the tasks so that the chronological constraints posed by the arcs are respected
  - The linear order may not be unique

Topological Sort Algorithm

1. Build an "indegree table" of the DAG
2. Output a vertex v with zero indegree
3. For vertex v, the arc (v, w) is no longer useful since the task (vertex) w does not need to wait for v to finish anymore
   - So after outputting the vertex v, we can remove v and all its outgoing arcs. The result graph is still a directed acyclic graph. So we can repeat from step 2 until no vertex is left

Demo

- demo-topological.ppt

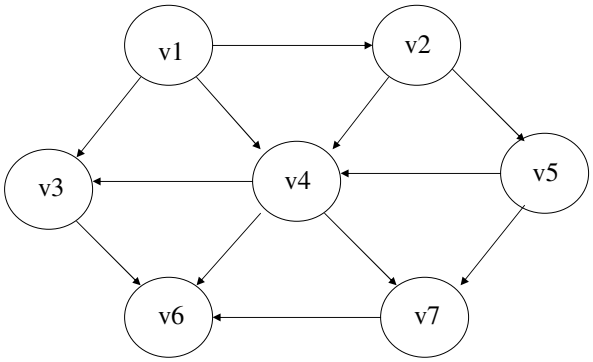
Topological Sort Algorithm

- The algorithm is very simple:

```
for i = 1 to V {
    Find any vertex with no incoming edges;
    Print this vertex, and remove it, along with
    its edges, from the graph;
}
```

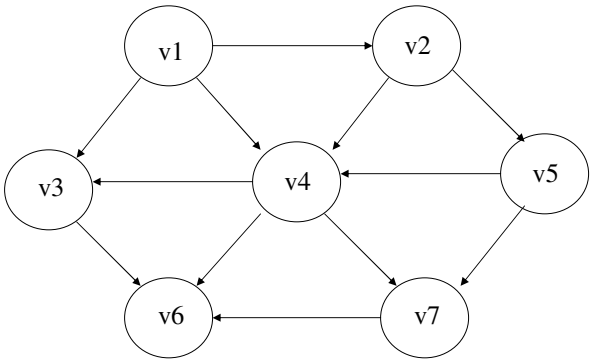
The for loop takes *O(V)* time and it also takes *O(V)* time to find a vertex with no incoming edges, so the total time is $O(V^2)$.

Example



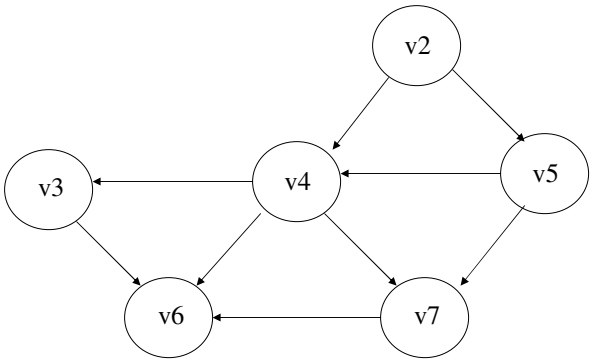
Two topological orderings are (v1, v2, v5, v4, v3, v7, v6) and (v1, v2, v5, v4, v7, v3, v6). Note there is no path from v3 to v7, or v7 to v3.

Example



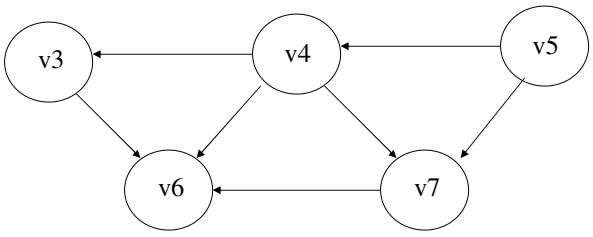
The vertex v1 has no incoming edges. Thus we have (v1) thus far for our topological sort. Now remove v1..

Example



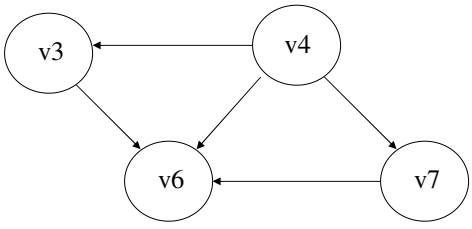
The vertex v2 has no incoming edges. Thus we have (v1, v2) thus far for our topological sort. Now remove v2..

Example



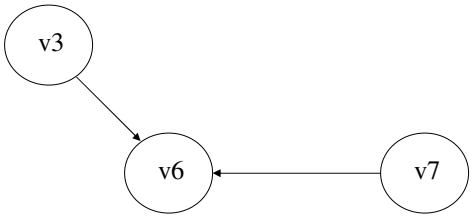
The vertex v5 has no incoming edges. Thus we have (v1, v2, v5) thus far for our topological sort. Now remove v5..

Example



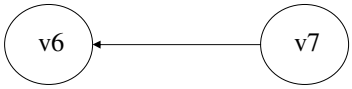
The vertex v4 has no incoming edges. Thus we have (v1, v2, v5, v4) thus far for our topological sort. Now remove v4..

Example



The vertices v3 and v7 has no incoming edges. Pick either (say v3). We have (v1, v2, v5, v4, v3) thus far for our topological sort. Now remove v3..

Example



The vertex v7 has no incoming edges. We have (v1, v2, v5, v4, v3, v7) thus far for our topological sort. Now remove v7..

Example

The vertex v6 has no incoming edges. We have (v1, v2, v5, v4, v3, v7, v6) for our topological sort. Done! Try this with the CS prerequisite graph as well.

Pseudocode

```
Algorithm TSort(G)
Input: a directed acyclic graph G
Output: a topological ordering of vertices
Initialize Q to be an empty queue;
For each vertex v
     do if indegree(v) = 0
             then enqueue(Q,v);
While Q is non-empty
     do v := dequeue(Q);
          output v;
          for each arc(v,w)
               do indegree(w) = indegree(w)-1;
                    if indegree(w) = 0
                         then enqueue(w);
```

Quiz 2

- Let a file describe the perquisites between classes as the following

```
CLASS CS140
  PREREQ CS102
CLASS CS160
  PREREQ CS102
CLASS CS302
  PREREQ CS140
CLASS CS311
  PREREQ MATH300
  PREREQ CS302
```

- Use the last graph API to write a program to give a topological order of these classes