Artificial Intelligence

For HEDSPI Project

Lecturer 9 – Propositional Logic

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Knowledge-based Agents

- Know about the world
  - They maintain a collection of facts (sentences) about the world, their Knowledge Base, expressed in some formal language.
- Reason about the world
  - They are able to derive new facts from those in the KB using some inference mechanism.
- Act upon the world
  - They map percepts to actions by querying and updating the KB.

What is Logic ?

- A logic is a triplet <L,S,R>
  - L, the language of the logic, is a class of sentences described by a precise syntax, usually a formal grammar
  - S, the logic's semantic, describes the meaning of elements in L
  - R, the logic's inference system, consisting of derivation rules over L
- Examples of logics:
  - Propositional, First Order, Higher Order, Temporal, Fuzzy, Modal, Linear, …

Propositional Logic

- Propositional Logic is about facts in the world that are either true or false, nothing else
- Propositional variables stand for basic facts
- Sentences are made of
  - propositional variables (A,B,…),
  - logical constants (TRUE, FALSE), and
  - logical connectives (not,and,or,..)
- The meaning of sentences ranges over the Boolean values {True, False}
  - Examples: It's sunny, John is married

Language of Propositional Logic

- Symbols
  - Propositional variables: A,B,…,P,Q,…
  - Logical constants: TRUE, FALSE
  - Logical connectives:
    ¬ , ∧ , ∨ , ⇒ , ⇔
- Sentences
  - Each propositional variable is a sentence
  - Each logical constant is a sentence
  - If ~ and are sentences then the following are sentences
    (~),¬~,~ ∧ ,~ ∨ ,~ ⇒ ,~ ⇔

Formal Language of Propositional Logic

- Formal Grammar
  - Sentence -> Asentence | Csentence
  - Asentence -> TRUE | FALSE | A | B|…
  - Csentence -> (Sentence) | Sentence | Sentence Connective Sentence
  - Connective -> ¬ , ∧ , ∨ , ⇒ , ⇔

Semantic of Propositional Logic

- The meaning of TRUE is always True, the meaning of FALSE is always False
- The meaning of a propositional variable is either True or False
  - depends on the interpretation
    - assignment of Boolean values to propositional variables
- The meaning of a sentence is either True or False
  - depends on the interpretation

Semantic of Propositional Logic

- True table

P	Q	Not P	P and Q	P or Q	P implies Q	P equiv Q
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

a⇒b ⇔ ¬a∨b ⇔ ¬b⇒¬a

Semantic of Propositional Logic

- Entailment
  - Given
    - A set of sentences S
    - A sentence A
  - We write

    S ╞ A

    if and only if every interpretation that makes all sentences in S true also makes A true
  - We said that S entails A

Inference in Propositional Logic

- Forward Chaining
- Backward Chaining

Forward Chaining

- Given a set of rules, i.e. formulae of the form

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \Rightarrow q$$

  and a set of known facts, i.e., formulae of the form

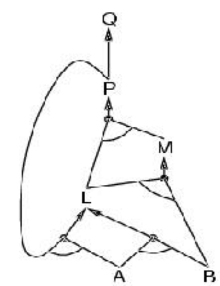
$$q, r, \ldots$$

- A new fact p is added
- Find all rules that have p as a premise
- If the other premises are already known to hold then
  - add the consequent to the set of know facts, and
  - trigger further inferences

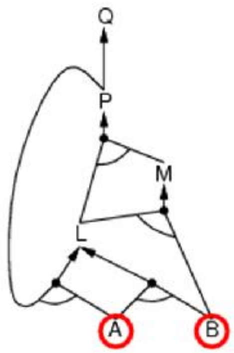
Forward Chaining

- Example

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



Forward Chaining



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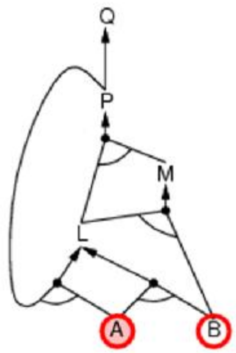

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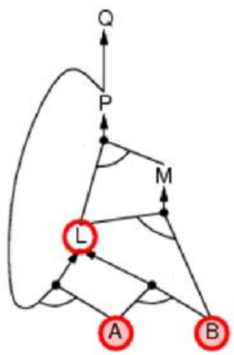

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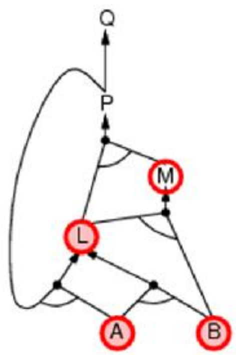

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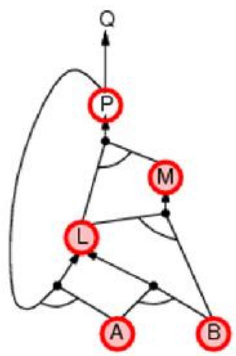

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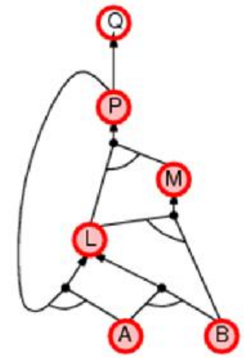


Forward Chaining

Input:

- Sentences/clauses in Horn format (Fact)
- A rule set R $p_1 \wedge p_2 \wedge ... \wedge p_n \Rightarrow q$
- Goal

Output:

- "Success" if Goal can be infered from Fact

Method: Use

- Temp - a set of propositional variables which are true at the current time
- Sat - a set of satisfied rules

Forward Chaining

$\{_1$ Temp = Fact;
Sat= FindRules(Temp,R);
while Sat<>0 and Goal $\not\subseteq$ Temp do
$\{_2$ r $\leftarrow$ get(Sat); /* r: left $\rightarrow$ q */
R = R \ {r}; Trace = Trace $\cup$ {r};
Temp = Temp $\cup$ {q};
Sat = FindRules(Temp,R)
$\}_2$
if Goal $\subseteq$ Temp then exit("Success")
else exit("Not success")
$\}_1$

Example

E1. Given Fact = $\{a,b,m_a\}$. Prove h_c

1. $a,b,m_a \rightarrow c$
2. $a,b,c \rightarrow A$
3. $b,A \rightarrow h_c$
4. $a,b,c \rightarrow B$
5. $a,b,c \rightarrow C$
6. $a,B \rightarrow h_c$
7. $A,B \rightarrow C$
8. $B,C \rightarrow A$
9. $A,C \rightarrow B$

Exercises

Compare stack and queue

1- Given Fact={a}, Goal={u}

1. $a \rightarrow b$
2. $b \rightarrow c$
3. $c \rightarrow d$
4. $a \rightarrow u$

2 - Given Fact={a}, Goal={u}

1. $a \rightarrow b$
2. $d \rightarrow c$
3. $c \rightarrow u$
4. $a \rightarrow m$
5. $b \rightarrow n$
6. $m \rightarrow p$
7. $p \rightarrow q$
8. $q \rightarrow u$

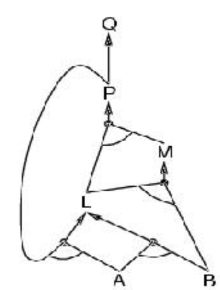
Backward Chaining

- Given a set of rules, and a set of known facts
- We ask whether a fact *P* is a consequence of the set of rules and the set of known facts
- The procedure check whether *P* is in the set of known facts
- Otherwise find all rules that have *P* as a consequent
  - If the premise is a conjunction, then process the conjunction conjunct by conjunct

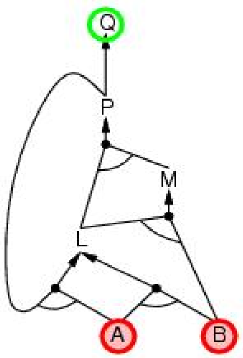
Backward Chaining

- Example

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



Backward Chaining



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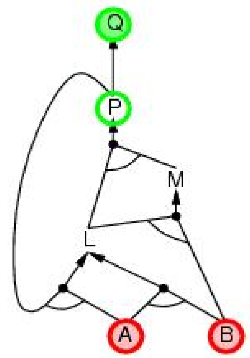

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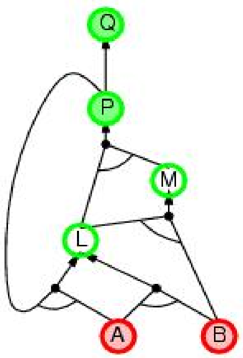

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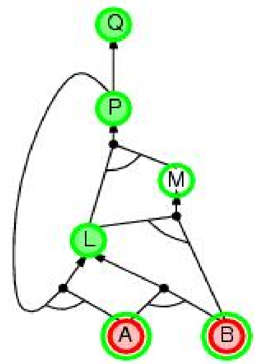
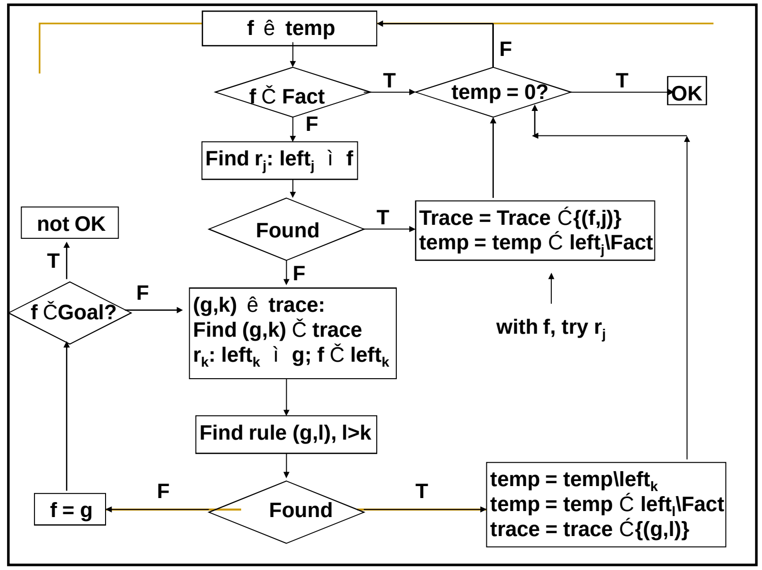


Backward Chaining

Variables:

- Goal: set of variables needed to be proved
- temp = {f| f is needed to be proved until now}
- trace ={(f,j)| to prove f, use rule j: $left_j \rightarrow f$}
- Flag Back = true when backtrack
  false otherwise



Backward Chaining

Example:

1. h_a,c → B
2. a,b,m_a →c
3. a,b,c → A
4. a,b,c → B
5. a,b,c → C
6. a,B→ h_c
7. b,A → h_c
8. c,S → h_c
9. a,b,c → S

Fact={a,b,m_a}; Goal={h_c}

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Exercises

E1. Given Fact={a,b,m_a}, Goal={h_c}

1. a,b,m_a → c
2. a,b,C → s
3. a,s → h_a
4. b,s → h_b
5. c,s → h_c
6. a,B → h_c
7. a,b,c → B

E2. Given Fact={a}, Goal={u}

1. a → b
2. d → c
3. c → u
4. a → m
5. b → n
6. m → p
7. p → q
8. q → u

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Transformation rules

$(\alpha \land \beta) \equiv (\beta \land \alpha)$
$(\alpha \lor \beta) \equiv (\beta \lor \alpha)$ } commutation

$((\alpha \land \beta) \land \gamma) \equiv (\alpha \land (\beta \land \gamma))$
$((\alpha \lor \beta) \lor \gamma) \equiv (\alpha \lor (\beta \lor \gamma))$ } combination

$\neg(\neg\alpha) \equiv \alpha$ double negation

$(\alpha \Rightarrow \beta) \equiv (\neg\beta \Rightarrow \neg\alpha)$ contrast

$(\alpha \Rightarrow \beta) \equiv (\neg\alpha \lor \beta)$

$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha))$

$\neg(\alpha \land \beta) \equiv (\neg\alpha \lor \neg\beta)$
$\neg(\alpha \lor \beta) \equiv (\neg\alpha \land \neg\beta)$ } de Morgan

$(\alpha \land (\beta \lor \gamma)) \equiv ((\alpha \land \beta) \lor (\alpha \land \gamma))$
$(\alpha \lor (\beta \land \gamma)) \equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma))$ } distribution

Transformation rules (con't)

Law of absorption:

- $(A \lor (A \land B) \equiv A$
- $(A \land (A \lor B)) \equiv A$

Rules about 0, 1:

- $A \land 0 \Leftrightarrow 0$
- $A \lor 1 \Leftrightarrow 1$
- $\neg 1 \Leftrightarrow 0$
- $A \lor 0 \Leftrightarrow A$
- $A \land 1 \Leftrightarrow A$
- $\neg 0 \Leftrightarrow 1$

Law of decentralization:

- $\neg A \lor A \Leftrightarrow 1$

Law of contradiction:

- $\neg A \land A \Leftrightarrow 0$

Transform into CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$

1. Remove $\Leftrightarrow$, replace $\alpha \Leftrightarrow \beta$ by $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$.
   $(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})$

2. Remove $\Rightarrow$, replace $\alpha \Rightarrow \beta$ by $\neg\alpha \lor \beta$.
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1})$

3. Move negation inward using the de Morgan's rule :
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})$

4. Applying the "and" distribution rule :
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$

Example

$(A \lor B) \rightarrow (C \rightarrow D)$

1. Remove $\Rightarrow$
   $\neg(A \lor B) \lor (\neg C \lor D)$
2. Move negation inward
   $(\neg A \land \neg B) \lor (\neg C \lor D)$
3. Distribution
   $(\neg A \lor \neg C \lor D) \land (\neg B \lor \neg C \lor D)$

Exercises

Transform the following expression into CNF.

1. $P \vee (\neg P \wedge Q \wedge R)$
2. $(\neg P \wedge Q) \vee (P \wedge \neg Q)$
3. $\neg(P \Rightarrow Q) \vee (P \vee Q)$
4. $(P \Rightarrow Q) \Rightarrow R$
5. $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \wedge S) \Rightarrow R)$
6. $(P \wedge (Q \Rightarrow R)) \Rightarrow S$
7. $P \wedge Q \Rightarrow R \wedge S$
8. $((a \vee b) \wedge c) \rightarrow (c \wedge d)$

Priority: $\neg \; \wedge \; \vee \; \rightarrow \; \leftrightarrow$

$$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$$
$$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$$
$$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$$
$$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$$
$$\neg(\neg \alpha) \equiv \alpha$$
$$(\alpha \Rightarrow \beta) \equiv (\neg \beta \Rightarrow \neg \alpha)$$
$$(\alpha \Rightarrow \beta) \equiv (\neg \alpha \vee \beta)$$
$$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$$
$$\neg(\alpha \wedge \beta) \equiv (\neg \alpha \vee \neg \beta)$$
$$\neg(\alpha \vee \beta) \equiv (\neg \alpha \wedge \neg \beta)$$
$$(\alpha \wedge (\beta \vee \gamma)) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$$
$$(\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$$

1. $P \vee (\neg P \wedge Q \wedge R)$
2. $(\neg P \wedge Q) \vee (P \wedge \neg Q)$
3. $\neg(P \Rightarrow Q) \vee (P \vee Q)$
4. $(P \Rightarrow Q) \Rightarrow R$
5. $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \wedge S) \Rightarrow R)$
6. $(P \wedge (Q \Rightarrow R)) \Rightarrow S$
7. $P \wedge Q \Rightarrow R \wedge S$
8. $((a \vee b) \wedge c) \rightarrow (c \wedge d)$

Resolution

Conjunctive Normal Form (CNF)
E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

- Resolution rule for CNF:

$$\frac{l_1 \vee \ldots \vee l_k, \qquad m_1 \vee \ldots \vee m_n}{l_1 \vee \ldots \vee l_{i-1} \vee l_{i+1} \vee \ldots \vee l_k \vee m_1 \vee \ldots \vee m_{j-1} \vee m_{j+1} \vee \ldots \vee m_n}$$

in which l_i and m_j are decentralized

E.g., $\dfrac{P_{1,3} \vee P_{2,2}, \quad \neg P_{2,2}}{P_{1,3}}$

Resolution

- Herbrand's Theorem (~1930)
  - A set of sentences S is unsatisfiable if and only there exists a finite subset S_g of the set of all ground instances Gr(S), which is unsatisfiabe
- Herbrand showed that there is a procedure to demonstrate the unsatisfiability of a unsatisfiable set of sentences
- Robinson propose the Resolution procedure (~1950)

Idea of Resolution

- Sun: Do you love me?
- Dad: yes, of course
- Sun: If you love me, then do you pamper me?
- Dad: yes
- Sun: If you pamper me, then if I ask you anything, will you give it to me?
- Dad: fine
- Sun: Give me money?
- Dad: No
- Sun: You are a liar

Idea of Resolution

- Refutation-based procedure
  - S |= A if and only if $S \cup \{\neg A\}$ is unsatisfible
- Resolution procedure
  - Transform $S \cup \{\neg A\}$ into a set of clauses
  - Apply Resolution rule to find a the empty clause (contradiction)
    - If the empty clause is found
      - Conclude S |= A
    - Otherwise
      - No conclusion

Idea of Resolution

- A clause is a disjunction of literals, i.e., has the form

$$P_1 \vee P_2 \vee \ldots \vee P_n \qquad P_i \equiv [\neg]R_i$$

- The empty clause corresponds to a contradiction
- Resolution rule

$$\frac{A \vee B \qquad \neg B \vee C}{A \vee C}$$

Robinson's Resolution

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  clauses ← the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  new ← { }
  loop do
    for each C_i, C_j **in** *clauses* **do**
      resolvents ← PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      new ← *new* ∪ *resolvents*
    if *new* ⊆ *clauses* **then return** *false*
    clauses ← *clauses* ∪ *new*

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Robinson's Resolution

Given KB = {P1, P2, ..., Pn}. Prove Q.
Add ¬Q to KB: KB = KB ∧ ¬Q. Prove unsatisfied.

1. Write each Pi, ¬Q in one line.
2. Transfer to CNF representation
   $(a_1 \vee \ldots \vee a_n) \wedge (b_1 \vee \ldots \vee b_n)$ (*)
3. Rewrite each line (*) into smaller lines:
   $a_1 \vee \ldots \vee a_n$
   $b_1 \vee \ldots \vee b_n$

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Robinson's Resolution

Consider 2 lines

u) $\neg p \vee q$

v) $p \vee r$

Resolution:

w) $q \vee r$

Contrast appears when KB contains 2 lines:

i) $\neg t$

ii) t

⇒ done

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Examples

E1)

1. a
2. $a \to b$
3. $b \to (c \to d)$
4. c

Prove d

E2)

1. $a \wedge b \to c$
2. $b \wedge c \to d$
3. a
4. b

Prove d

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Examples

E3)

1. p
2. $p \to q$
3. $q \wedge r \wedge s \to t$
4. $p \to u$
5. $v \to w$
6. $u \to v$
7. $v \to t$

Given r,s are true. Prove t

E4)

1. $((a \vee b) \wedge c) \to (c \wedge d)$
2. $a \wedge m \wedge d \to f$
3. $m \to b \wedge c$
4. $a \to c$
5. $(a \wedge f) \to (\neg e \vee g)$
6. $(m \wedge f) \to g$

Given a,m are true. Prove g

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Exercise 5

1. $a1 \vee a2 \Rightarrow a3 \vee a4$
2. $a1 \Rightarrow a5$
3. $a2 \wedge a3 \Rightarrow a5$
4. $a2 \wedge a4 \Rightarrow a6 \wedge a7$
5. $a5 \Rightarrow a7$
6. $a1 \wedge a3 \Rightarrow a6 \vee a7$

- Given a1, a2 are true .
- Transfer the above sentences to the CNF representation
- Apply the Robinson's resolution, prove a7 is true.

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Exercise 6

Given

1. c
2. d
3. $b \wedge d \Rightarrow g$
4. $b \wedge c \Rightarrow e$
5. $d \Rightarrow e$
6. $(b \wedge d) \Rightarrow h$
7. $a \wedge c \Rightarrow f \wedge g$
8. $d \vee c \Rightarrow a \vee b$
9. $c \wedge e \Rightarrow g$

- Transfer the above sentences to the CNF representation
- Apply the Robinson's resolution, prove $e \wedge g$ true

Exercise 7

1. $(a \wedge b) \Rightarrow c$
2. $(a \vee b) \Rightarrow (\neg c \vee d)$
3. $(m \wedge u) \Rightarrow c$
4. $a \Rightarrow (g \wedge f)$
5. $(x \wedge b) \Rightarrow ((d \vee m) \wedge (g \vee u))$
6. $((d \wedge e) \text{ v } a) \Rightarrow (g \text{ v } u)$
7. $(g \wedge u) \Rightarrow (y \wedge d)$
8. $d \Rightarrow e$

- Transfer the above sentences to the CNF representation
- Given a, b true. Apply the Robinson's resolution, prove e is true